

Finite Heisenberg Groups in Quiver Gauge Theories

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Abstract

We show by direct construction that a large class of quiver gauge theories admits actions of finite Heisenberg groups. We consider various quiver gauge theories that arise as AdS/CFT duals of orbifolds of \mathbb{C}^3 , the conifold and its orbifolds and some orbifolds of the cone over $Y^{p,q}$. Matching the gauge theory analysis with string theory on the corresponding spaces implies that the operators counting wrapped branes do not commute in the presence of flux.

1 Introduction

The AdS/CFT has generally been used to obtain information about strongly coupled gauged theories using the weakly coupled supergravity side [1]. Given that our technology for understanding string theory in backgrounds with Ramond-Ramond fluxes is still inadequate it could be advantageous to use gauge theories to understand fundamental properties of string theory such as the nature of D branes. In fact, in a very interesting paper Gukov, Rangamani and Witten did just that [2]. They matched operators on the field theory defined as a \mathbb{Z}_3 orbifold of $\mathcal{N} = 4$ super Yang Mills to wrapped branes in type IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_3$ with five-form flux, therefore uncovering a very interesting noncommutative structure. They found a finite Heisenberg group realized by discrete transformations in the gauge theory. These transformations can subsequently be interpreted as operators counting various wrapped branes in the string theory.

The idea that D-brane charge in the presence of flux might not be a commutative quantity has far reaching implications for our understanding of D-branes. It has recently been argued by D. Belov, G. Moore and others [3] that this is, in fact, the general situation whenever the space has homology groups with nontrivial torsion classes. An interesting review of the mathematical background can be found in [4].

If the structure uncovered in [2] is generic for D-branes in backgrounds with RR fluxes then we should be able to display it in a more general setting. With this motivation in mind we turned to generalizations of the construction of [2]. We show by direct construction that a large class of quiver gauge theories admits an action of the Heisenberg group. In particular we consider various quiver gauge theories that arise as duals in the AdS/CFT sense of orbifolds of \mathbb{C}^3 , the conifold and its orbifolds and some orbifolds of the cone over $Y^{p,q}$.

Our main result can be formulated as follows. For a large class of quiver gauge theories with gauge group $SU(N)^p$, there is a set of discrete transformations A, B and C satisfying

$$A^q = B^q = C^q = 1, \quad AB = BAC. \quad (1.1)$$

where q is some integer number which depends on the particular structure of the quiver. These transformations satisfy three important properties: (i) leave the superpotential invariant, (ii) satisfy the anomaly cancelation for all $SU(N)$ gauge groups, and (iii) the above group relations are true up to elements in the center of the gauge group

$SU(N)^p$, that is, up to gauge transformations.

This structure can be interpreted in terms of D branes in the dual string theory. In this case the operators A, B and C count the number of wrapped fundamental strings, D-strings and D3 branes respectively. The matching is impressive, in particular the number q above is related to torsion classes in the third homology group $H_3(X, \mathbb{Z}) = \mathbb{Z}_q$, where X is the horizon manifold of the space dual to the quiver gauge theory.

The organization of this note is as follows. In section two we present the guiding principles in the search for the set of symmetries realizing finite Heisenberg groups in some quiver gauge theories. After the general setup we consider orbifolds of $\mathcal{N} = 4$ SYM, these theories can be understood in the AdS/CFT sense as orbifolds of \mathbb{C}^3 ; we also discuss the conifold and its orbifolds and some orbifolds of the cone over $Y^{p,q}$. Section three contains comments on the D-brane interpretation. We conclude in section four.

2 Finite Heisenberg groups in quiver gauge theories

In this section we discuss the set up of the general construction. We want to display how the constraints of classical invariance of the superpotential and anomaly cancelation determine the symmetries.

As mentioned in the introduction, we seek discrete transformations A, B and C of the gauge theory. First, these transformations are discrete and cyclical:

$$A^q = B^q = C^q = 1, \tag{2.1}$$

where q is an integer that depends on the concrete theory. More interestingly, these transformations satisfy a finite Heisenberg group relation

$$AB = BAC, \tag{2.2}$$

and the element C commutes with A and B .

We take a constructive approach. As a first step we consider a special class of quiver diagrams that admit a shift symmetry. This means that there is an obvious map of fields to fields and gauge groups to gauge groups. In the case of a \mathbb{Z}_q orbifold of $\mathcal{N} = 4$ SYM one has q gauge groups. The A symmetry is a cyclic permutation of the gauge groups. There is an alternative way to view the appearance of the A transformation from the dual string theory side. For example, the correspondent dual

is described by N D3 branes near a $\mathbb{C}^3/\mathbb{Z}_q$ singularity. In the string theory setup there is a natural shift symmetry. The shift symmetry is just the action of moving the stack of branes onto their image (and so this symmetry is a \mathbb{Z}_q). The operators B and C are then realized as “rephasing” symmetries which multiply the superfields, component by component, by a constant phase.

To begin discussion of these symmetries, we set our labeling conventions. We will label vertices with numbers. Superfields will then be labeled with an upper case letter carrying a subscript that denotes which vertex it originates from (or goes to, depending on the case). If there is a global symmetry under which a set of fields transforms, then this index will be labeled by a superscript. The phase that we shift a field by will be denoted by lowercase letter and the same subscript that labels the field. For example, in a quiver diagram with a global $SU(2)$ symmetry, the fields will be labeled

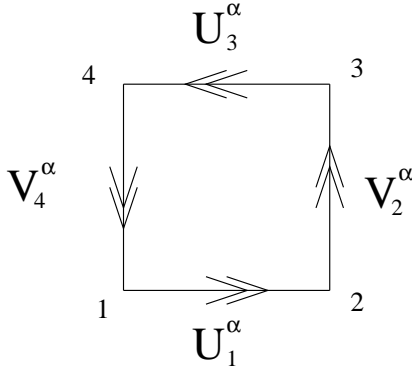


Figure 1: Example of a quiver diagram with global symmetries.

where the U_n^α and V_n^α transform as doublets under the $SU(2)$. To preserve a global symmetry we must scale all fields in a representation by the same phase, e.g. $U_1^\alpha \mapsto u_1 U_1^\alpha$. We again emphasize that this rephasing of the fields to act on the component fields separately, and so we are not combining this with an R symmetry associated with the Grassmann variables.

For this to be a classical symmetry, we require invariance of the superpotential. In general, one can read the superpotential terms by going in loops in a diagram, making sure that the fields can be combined into invariant terms under any global symmetry. We will only consider loops that include 3 and 4 fields. For example, in figure (1) there is only one superpotential term $\propto \epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'} \text{Tr} (U_1^\alpha V_2^{\alpha'} U_3^\beta V_4^{\beta'})$. Invariance of the superpotential implies that we must demand that classically $u_1 v_2 u_3 v_4 = 1$. This generalizes easily to other quivers. For a monomial term in a superpotential one

replaces the fields by the associated scalings and requires this product to be 1:

$$\mathcal{W} = \cdots + g \text{Tr} (U_1 U_2 \cdots U_k) + \cdots \longrightarrow u_1 u_2 \cdots u_k = 1, \quad \cdots \quad (2.3)$$

Now we look for those symmetries that are not broken by quantum effects. The problematic terms in the path integral measure are the fermions. The basic point is that in instanton backgrounds, the Dirac operator has fermion zero modes while the conjugate Dirac operator does not. This leads to an asymmetry in the path integral measure, and so the above scalings will, in general, be anomalous. The number of fermion zero modes in a background with instanton number J is $2T(\mathbf{r}) \times J$. One can see this dependence on $T(\mathbf{r})$ coming from the $(j_{phase}^\mu, A_{SU(N)}, A_{SU(N)})$ triangle anomaly. The most restrictive case for us is when $J = 1$ because this will raise the scaling to the smallest power. Let us take an example where the $SU(N)$ associated with vertex 1 has instanton number 1. To consider the anomaly, we consider the part of the diagram around this $SU(N)$:

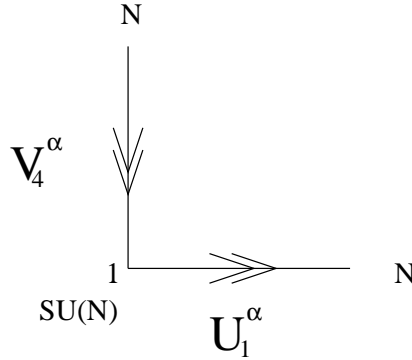


Figure 2: Anomaly contribution for a $SU(N)$ factor in the working example.

Note that the end of the arrows have an N fold degeneracy due to the $SU(N)$ gauge groups not under consideration. Therefore, there are $N \times 2 \times 2T(\mathbf{r})$ fermion zero modes associated with each of the fields U_1 and V_4 . These are (anti) fundamentals, and so $T(\mathbf{r}) = 1/2$. We conclude that the measure transforms unless $(v_4^2 u_1^2)^N = 1$. One may repeat this calculation for the other vertices, and one finds that $(u_i^2 v_{i\pm 1}^2)^N$ for $i = 1, 3$ and the subscripts are to be read mod 4. For a general diagram one considers one gauge group at a time, and considers the fields that couple to this gauge group. Then, one counts the multiplicity of the fields by the number of arrows times the rank of the gauge group at the other end of the arrow. The condition that this is a symmetry is that $\prod a_i^{(mult.)_i} = 1$. For example, a vertex in a quiver diagram

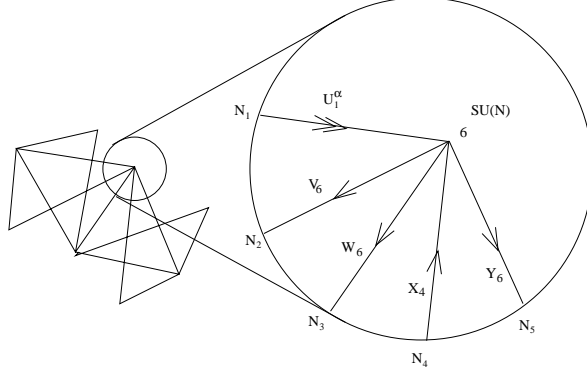


Figure 3: Anomaly contribution for a $SU(N)$ factor in the general case.

$$u_1^{2N_1} v_6^{N_2} w_6^{N_3} x_4^{N_4} y_6^{N_5}. \quad (2.4)$$

We now turn to a particular case of the above symmetries that are related to the gauge symmetry. We will consider only quiver theories that have an $SU(N)$ at every vertex. For quiver diagrams with n vertices the gauge group is then $SU(N)^n$ with center $(\mathbb{Z}_N)^n$. The center of the gauge group corresponds to scaling each field by an N^{th} root of unity, but in such a way as to leave the superpotential invariant. These then fall into the category just described. The gauge symmetry is a redundancy, and so we identify the scalings described in the last paragraph up to elements of the center of the gauge group.

2.1 Orbifolds of \mathbb{C}^3

The gauge theories obtained by orbifolding $\mathcal{N} = 4$ were discussed in the context of the AdS/CFT by [5, 6]. The techniques were essentially developed by Douglas and Moore in [7].

Let us first discuss the action of the orbifold on \mathbb{C}^3 . Let the complex coordinate on \mathbb{C}^3 be z_1, z_2 and z_3 , the orbifold of \mathbb{Z}_p acts as

$$(z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, \omega^{-2} z_3), \quad (2.5)$$

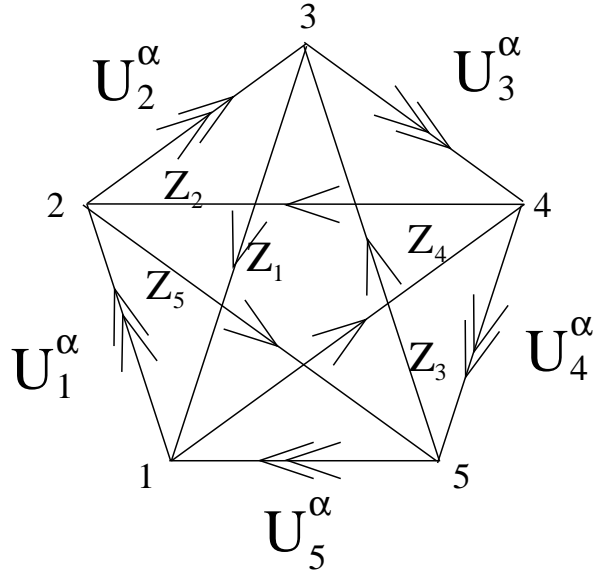
where ω is a p -th root of unity. It is important that this transformation preserves the natural holomorphic $(3, 0)$ -form that determines the Calabi-Yau structure, that is, it leaves $dz_1 \wedge dz_2 \wedge dz_3$ invariant.

Let us consider the case \mathbb{Z}_5 for concreteness but the techniques generalize directly. The general idea for orbifolding a theory consist on working in its covering space. In the case of $SU(N)$ gauge group therefore consider a theory with gauge group $SU(N)^5$.

The action of the orbifold treats coordinates z_1 and z_2 identically, in the quiver diagram this implies that we have a doublet under $SU(2)$ which we denote by U_i^α , where α is the $SU(2)$ index.

The superpotential terms can be obtained by traveling in loops in the quiver diagram.

$$W = \epsilon_{\alpha\beta} U_1^\alpha U_2^\beta Z_1 - \epsilon_{\alpha\beta} U_2^\alpha U_3^\beta Z_2 + \epsilon_{\alpha\beta} U_3^\alpha U_4^\beta Z_3 - \epsilon_{\alpha\beta} U_4^\alpha U_5^\beta Z_4 + \epsilon_{\alpha\beta} U_5^\alpha U_1^\beta Z_5. \quad (2.6)$$



The A symmetry is easy to spot in this case:

$$A : \begin{pmatrix} U_i^\alpha \\ Z_i \end{pmatrix} \mapsto \begin{pmatrix} U_{i+1}^\alpha \\ Z_{i+1} \end{pmatrix}. \quad (2.7)$$

This operation obviously has the property $A^5 = 1$.

We now proceed to look for rephasing of the fields that leave the theory invariant. The general transformation, as explained at the beginning of this section involves: $U_i \mapsto u_i U_i$ and $Z_i \mapsto z_i Z_i$, where u_i and z_i are some root of unity. Invariance of the superpotential implies an interesting factorization of the problem. For example, for the first term in the superpotential (the loop starting in the lower left corner) we have that

$$z_1 u_1 u_2 = 1. \quad (2.8)$$

We view this, and similar equalities as determining the scaling of the Z fields once the scalings of the U fields is known. The anomaly condition for the lower left corner gives:

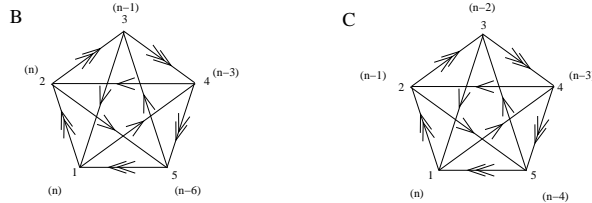
$$(u_5^2 z_4 z_1 u_1^2)^N = 1, \quad (2.9)$$

we can further eliminate the z' s to obtain expressions of the form:

$$\left(\frac{u_1 u_5}{u_2 u_4} \right)^N = 1. \quad (2.10)$$

Similar expressions are obtained with a cyclical reordering of the indices. Note that an interesting pattern arises: product of two consecutive phases equals the product of the phase before and after the sequence. We could now look for solutions to the above equations and find the B and C transformation. Here we will follow a different path that automatically eliminates some issues with transformations in the center of the gauge group.

To identify the scaling symmetries, we find it convenient to consider certain members of the center of the gauge group. To do so, we associate an integer n_i with each vertex. This integer tells which member of the center of that gauge group we are rephasing by: $e^{\frac{2\pi i n_i}{N}} \mathbb{I}_{N \times N}$. This then gives us a prescription for how to rephase the fields in a gauge invariant way. We consider the two scalings given by



which translate to

$$B : \begin{aligned} (U_1, U_2, U_3, U_4, U_5) &\mapsto (U_1, \gamma U_2, \gamma^2 U_3, \gamma^3 U_4, \gamma^{-6} U_5) \\ (Z_1, Z_2, Z_3, Z_4, Z_5) &\mapsto (\gamma^{-1} Z_1, \gamma^{-3} Z_2, \gamma^{-5} Z_3, \gamma^3 Z_4, \gamma^6 Z_5) \end{aligned} \quad (2.11)$$

$$C : \begin{aligned} (U_1, U_2, U_3, U_4, U_5) &\mapsto (\gamma U_1, \gamma U_2, \gamma U_3, \gamma U_4, \gamma^{-4} U_5) \\ (Z_1, Z_2, Z_3, Z_4, Z_5) &\mapsto (\gamma^{-2} Z_1, \gamma^{-2} Z_2, \gamma^{-2} Z_3, \gamma^3 Z_4, \gamma^3 Z_5) \end{aligned} \quad (2.12)$$

for the fields. Note that the above transformations were picked to have B as close to a “clock” symmetry, and C was designed “undo” a shift of the above fields. The structure of the Z field transformations come as a consequence of the U fields. Finally, the above transformations are a redundancy because $\gamma^N = 1$, i.e. is in the center of

the gauge group. However, to find non trivial elements, we wish to promote this action so that it is not in the center of the gauge group. We consider instead

$$B : \begin{aligned} (U_1, U_2, U_3, U_4, U_5) &\mapsto (U_1, \omega U_2, \omega^2 U_3, \omega^3 U_4, \omega^{-6} U_5) \\ (Z_1, Z_2, Z_3, Z_4, Z_5) &\mapsto (\omega^{-1} Z_1, \omega^{-3} Z_2, \omega^{-5} Z_3, \omega^3 Z_4, \omega^6 Z_5) \end{aligned} \quad (2.13)$$

$$C : \begin{aligned} (U_1, U_2, U_3, U_4, U_5) &\mapsto (\omega U_1, \omega U_2, \omega U_3, \omega U_4, \omega^{-4} U_5) \\ (Z_1, Z_2, Z_3, Z_4, Z_5) &\mapsto (\omega^{-2} Z_1, \omega^{-2} Z_2, \omega^{-2} Z_3, \omega^3 Z_4, \omega^3 Z_5) \end{aligned} \quad (2.14)$$

Considering the anomaly condition at the vertices, we conclude that the above transformations are anomaly free if $\omega^{5N} = 1$. For example, considering node 1, the fermion measure transforms as

$$\omega^{2(-4)N} \omega^{2N} \omega^{-2N} \omega^{3N} = \omega^{-5N} \quad (2.15)$$

under transformation C , and

$$\omega^{2(-6)N} \omega^{3N} \omega^{-N} = \omega^{-10N} \quad (2.16)$$

under B . We demand these to be 1. This gives us immediately that we take $B^5 = C^5 = 1$ because these are in the center of the gauge group.

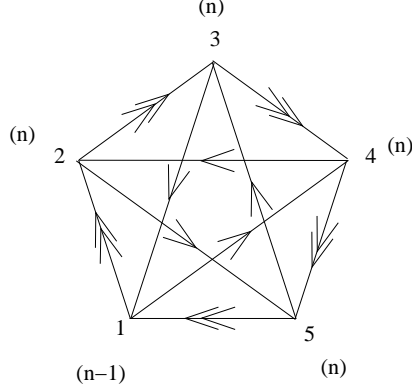
One may also check that in the U sector

$$A_U C_U = C_U A_U \times \begin{pmatrix} \omega^5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega^{-5} \end{pmatrix} \equiv C_U A_U \times M_U \quad (2.17)$$

and that in the Z sector

$$A_Z C_Z = C_Z A_Z \times \begin{pmatrix} \omega^{-5} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega^5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \equiv C_Z A_Z \times M_Z. \quad (2.18)$$

The matrices above are in the center of the gauge group with



and $\gamma = \omega^5$. It is also easy to check that

$$\begin{aligned} A_U B_U &= B_U A_U C_U \times M_U \\ A_Z B_Z &= B_Z A_Z C_Z \times M_Z. \end{aligned} \quad (2.19)$$

The extra factors on the right are in fact part of the center of the gauge group as before. Therefore, we see that we have

$$AB = BAC, \quad AC = CA, \quad BC = CB, \quad A^5 = B^5 = C^5 = 1. \quad (2.20)$$

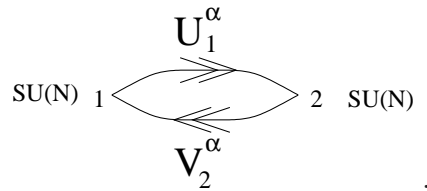
where the equalities are to be read only up to the center of the gauge group.

2.2 The conifold and its Orbifolds

The gauge theory of the worldvolume of N D3 branes in the conifold singularity was introduced by Klebanov and Witten [8]. It admits a holographic description in terms of IIB string theory on $AdS_5 \times T^{1,1}$. The theory has gauge group $SU(N) \times SU(N)$ and matter content described by two fields U^α and V^α which are doubles with respect to $SU(2)$ and transform in the bifundamental representation.

	$SU_1(N)$	$SU_2(N)$
U_1^α	N	\bar{N}
V_2^α	\bar{N}	N

A simple way to represent this theory is its quiver diagram



The superpotential of the theory is

$$W = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}U^\alpha V^\beta U^\gamma V^\delta. \quad (2.21)$$

A natural candidate for the A transformation is $A : (U, V) \mapsto (V, U)$.

We can start by considering arbitrary discrete transformation of the form

$$(U, V) \mapsto (uU, vV). \quad (2.22)$$

The condition that this transformation preserves the superpotential implies that

$$u^2 v^2 = 1. \quad (2.23)$$

The absence of anomalies implies that

$$u^{2N} v^{2N} = 1. \quad (2.24)$$

For ω a square root of unity we have.

$$A : (U, V) \mapsto (V, U), \quad (2.25)$$

$$B : (U, V) \mapsto (U, \omega V), \quad (2.26)$$

$$C : (U, V) \mapsto (\omega U, \omega^{-1} V). \quad (2.27)$$

It is easy to show that A, B and C satisfy the relations of a finite Heisenberg group with four elements.

There is, however, a problem with our transformation. The superpotential actually transforms by a minus sign. The fact that the superpotential transforms by a minus sign under $(U, V) \mapsto (V, U)$ was already pointed out in [8]. This transformation can, in principle, be accompanied by an R -symmetry transformation $\theta \mapsto i\theta$ which compensates. Thus we could conclude that the Heisenberg group of the conifold is anomalous unless it can be accompanied by an R -symmetry rotation. This is more like an accidental symmetry, we will comment on this in the next section.

2.2.1 \mathbb{Z}_2 orbifold of the conifold

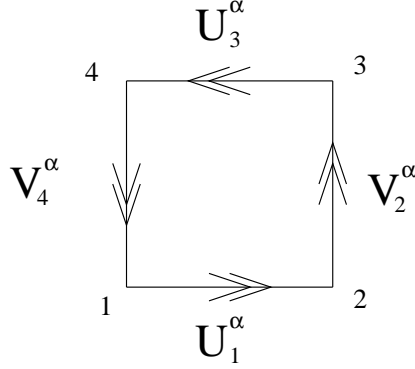
The gauge theory living in the worldvolume of N D3 branes placed at the singularity of the \mathbb{Z}_2 orbifold can be obtained from the previous construction. For the case of the \mathbb{Z}_2 orbifold we have that the superpotential is

$$W = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\text{Tr}U_1^\alpha V_2^\beta U_3^\gamma V_4^\delta. \quad (2.28)$$

The transformation rules for the fields are

	$SU_1(N)$	$SU_2(N)$	$SU_3(N)$	$SU_4(N)$
U_1^α	N	\bar{N}	1	1
V_2^α	1	N	\bar{N}	1
U_3^α	1	1	N	\bar{N}
V_4^α	\bar{N}	1	1	N

All the above information can also be read off from the quiver diagram



A candidate for the A transformation is

$$A : (U_1, V_2, U_3, V_4) \mapsto (U_3, V_4, U_1, V_2). \quad (2.29)$$

We now look for discrete transformations of the form

$$(U_1, V_2, U_3, V_4) \mapsto (u_1 U_1, v_2 V_2, u_3 U_3, v_4 V_4). \quad (2.30)$$

Invariance of the superpotential implies that

$$u_1 v_2 u_3 v_4 = 1. \quad (2.31)$$

Anomaly cancelation implies that

$$(u_1 v_2)^{2N} = (v_2 u_3)^{2N} = (u_3 v_4)^{2N} = (v_4 u_1)^{2N} = 1. \quad (2.32)$$

We find the following solutions for B and C , with the condition that $\omega^{2N} = 1$:

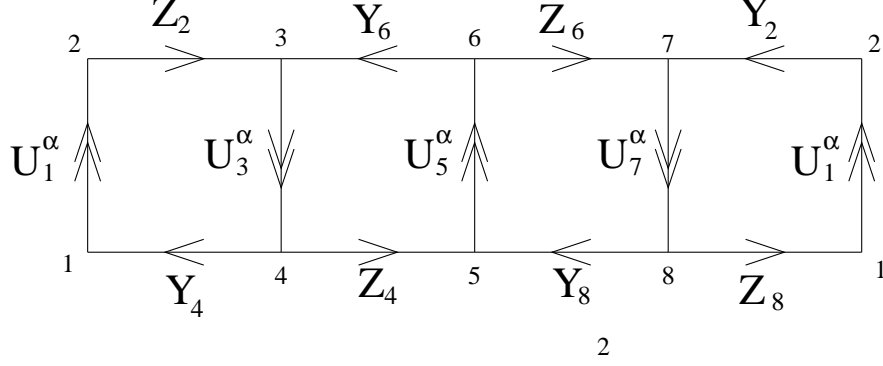
$$A : (U_1, V_2, U_3, V_4) \mapsto (U_3, V_4, U_1, V_2), \quad (2.33)$$

$$B : (U_1, V_2, U_3, V_4) \mapsto (\omega U_1, V_2, U_3, \omega^{-1} V_4), \quad (2.34)$$

$$C : (U_1, V_2, U_3, V_4) \mapsto (\omega U_1, \omega V_2, \omega^{-1} U_3, \omega^{-1} V_4) \quad (2.35)$$

2.2.2 \mathbb{Z}_4 orbifold of the conifold

Let us consider the quiver diagram corresponding to $Y^{4,0}$, which will capture all of the features of the general case.



Consider the A element generated by

$$\begin{aligned} (U_1, U_3, U_5, U_7) &\mapsto (U_7, U_1, U_3, U_5), \\ A : (Z_2, Z_4, Z_6, Z_8) &\mapsto (Z_8, Z_2, Z_4, Z_6), \\ (Y_2, Y_4, Y_6, Y_8) &\mapsto (Y_8, Y_2, Y_4, Y_6). \end{aligned} \quad (2.36)$$

Note that the most efficient way to represent this transformation is by action on all kinds of fields separately.

We can also think of the A transformation as a permutation on the nodes:

$$A : (1, 3, 5, 7) \mapsto (7, 1, 3, 5), \quad (2, 4, 6, 8) \mapsto (8, 2, 4, 6). \quad (2.37)$$

The superpotential for this theory is

$$W \sim \epsilon_{\alpha\beta} U_1^\alpha Z_2 U_3^\beta Y_4 - \epsilon_{\alpha\beta} U_3^\alpha Z_4 U_5^\beta Y_6 + \epsilon_{\alpha\beta} U_5^\alpha Z_6 U_7^\beta Y_8 - \epsilon_{\alpha\beta} U_7^\alpha Z_8 U_1^\beta Y_2. \quad (2.38)$$

Invariance of the superpotential implies that

$$u_1 z_2 u_3 y_4 = 1, \quad u_3 z_4 u_5 y_6 = 1, \quad u_5 z_6 u_7 y_8 = 1, \quad u_7 z_8 u_1 y_2 = 1. \quad (2.39)$$

with an ω such that $\omega^{4N} = 1$ we find that the remaining transformations are:

$$\begin{aligned} (U_1, U_3, U_5, U_7) &\mapsto (U_1, \omega U_3, \omega^2 U_5, \omega^3 U_7), \\ B : (Z_2, Z_4, Z_6, Z_8) &\mapsto \omega^{-3}(\omega^3 Z_2, \omega^2 Z_4, \omega Z_6, Z_8), \\ (Y_2, Y_4, Y_6, Y_8) &\mapsto \omega^{-3}(\omega^3 Y_2, \omega^2 Y_4, \omega Y_6, Y_8). \end{aligned} \quad (2.40)$$

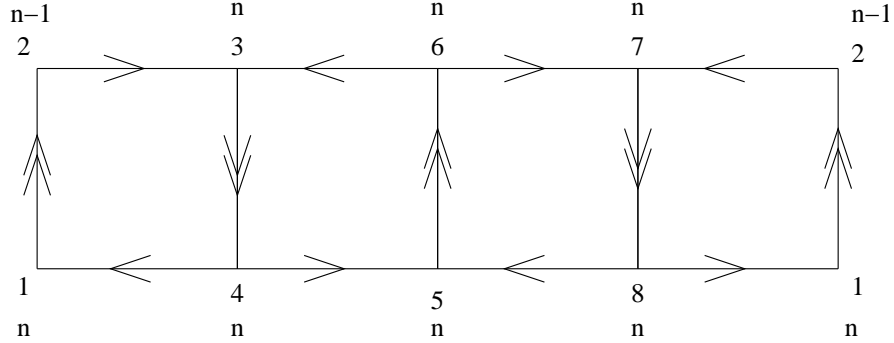
$$\begin{aligned}
(U_1, U_3, U_5, U_7) &\mapsto (\omega U_1, \omega U_3, \omega U_5, \omega U_7), \\
C : (Z_2, Z_4, Z_6, Z_8) &\mapsto (\omega^{-1} Z_2, \omega^{-1} Z_4, \omega^{-1} Z_6, \omega^{-1} Z_8), \\
(Y_2, Y_4, Y_6, Y_8) &\mapsto (\omega^{-1} Y_2, \omega^{-1} Y_4, \omega^{-1} Y_6, \omega^{-1} Y_8).
\end{aligned} \tag{2.41}$$

Note that B and C are given as transformation acting on the three sets of fields that we have, namely, U, Y and Z .

Now we will check explicitly that the above transformations satisfy the Heisenberg group laws. First, let us note that we will have to use the center of the gauge group to complete the transformations. The matrix representations of the above transformations are block diagonal in the U, Y and Z fields. One finds that these matrices obey

$$AB = BAC \times \begin{pmatrix} \omega^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{I}_{3 \times 3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^{-4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{3 \times 3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{I}_{3 \times 3} \end{pmatrix}. \tag{2.42}$$

The last matrix on the right hand side is a member of the center of the gauge group. First note that because ω is a $4N^{\text{th}}$ root of unity, that ω^4 is an N^{th} root of unity. Therefore to each gauge group we associate a ω^{4n_i} , and rephase the fields according to whether they are fundamental or antifundamental. The center of the gauge group corresponding to the above matrix is then given by



for any integer n , and as before the N^{th} root of unity associated with the center of the gauge group is $\gamma = \omega^4$. The clock symmetry can shift this action to any of the other sets of fields, acting on $(1, 2), (3, 4), (5, 6)$ and $(7, 8)$ fields independently. We also see that any of these operations applied 4 times is gauge equivalent to the identity. We

therefore find that

$$A^4 = B^4 = C^4 = 1, \quad AB = BAC. \quad (2.43)$$

2.2.3 \mathbb{Z}_p orbifold, the general case

The procedure of the last subsection is easily lifted to the general $Y^{(p,0)}$ case. The transformations are

$$A : \begin{aligned} (1, 3, 5, \dots, 2p-3, 2p-1) &\mapsto (2p-1, 1, 3, \dots, 2p-5, 2p-3), \\ (2, 4, 6, \dots, 2p-2, 2p) &\mapsto (2p, 2, 4, \dots, 2p-4, 2p-2). \end{aligned} \quad (2.44)$$

$$B : \begin{aligned} (U_1, U_3, \dots, U_{2p-1}) &\mapsto (U_1, \omega U_3, \omega^2 U_5, \dots, \omega^{p-1} U_{2p-1}), \\ (Z_2, Z_4, \dots, Z_{2p}) &\mapsto (1 Z_2, \omega^{-1} Z_4, \dots, \omega^{-(p-1)} Z_{2p}), \\ (Y_2, Y_4, \dots, Y_{2p}) &\mapsto (1 Y_2, \omega^{-1} Y_4, \dots, \omega^{-(p-1)} Y_{2p}). \end{aligned} \quad (2.45)$$

$$C : \begin{aligned} (U_1, U_3, \dots, U_{2p-1}) &\mapsto (\omega U_1, \omega U_3, \dots, \omega U_{2p-1}), \\ (Z_2, Z_4, \dots, Z_{2p}) &\mapsto (\omega^{-1} Z_2, \omega^{-1} Z_4, \dots, \omega^{-1} Z_{2p}), \\ (Y_2, Y_4, \dots, Y_{2p}) &\mapsto (\omega^{-1} Y_2, \omega^{-1} Y_4, \dots, \omega^{-1} Y_{2p}) \end{aligned} \quad (2.46)$$

$$\omega^{pN} = 1. \quad (2.47)$$

These operations satisfy

$$A^p = B^p = C^p = 1, \quad AB = BAC \quad (2.48)$$

up to the center of the gauge group. The element of the center of the gauge group discussed earlier is easily generalized: promote $4 \rightarrow p$ and all new gauge groups are assigned pn .

2.3 Orbifolds of $Y^{p,q}$

Another very interesting class of gauge theories are the quiver gauge theories obtained as the gauge theory dual of string theory on $AdS_5 \times Y^{p,q}$ with 5-form flux. A very complete discussion of $Y^{p,q}$ spaces is presented in [9]. The field theory aspects are presented in [10–12].

We now display some general results for orbifolds of $Y^{p,q}$, specifically those that have p and q not relatively prime (this has the effect of changing the periodicity of the β angle, and so is an abelian orbifold in this angle). We will call the greatest common divisor of p and q $GCD(p, q) = s$. We will follow the notation introduced

in [13], where an arbitrary $Y^{p,q}$ quiver gauge theory is constructed as a sequence of two primitive elements. The quiver diagram that we associate with this geometry is that of $Y^{p/s,q/s}$ repeated s times:

$$\underbrace{(\underbrace{\sigma\tilde{\sigma}\tau\cdots\cdots}_{((p-q)/a)\ \tau\text{-type}, (q/a)\ \sigma\text{-type}})(\cdots)(\cdots)\cdots}_{s\text{-times}}. \quad (2.49)$$

where σ and τ are the unit cells

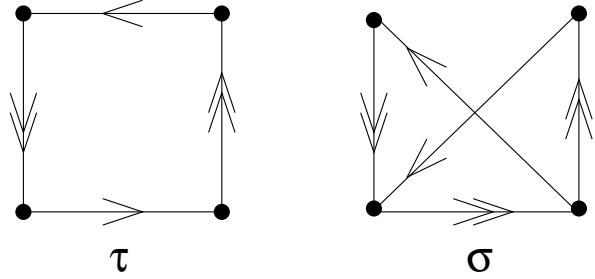


Figure 4: The unit cells τ and σ .

For details of constructing the quivers gauge theories for $Y^{p,q}$ we refer the reader to [11, 13]. Because the ends are identified one may rearrange the above diagrams into polygon diagrams which have $2p$ sides, and a number of internal line, which depends on the particular value of q . The A transformation is then cyclicly mapping the primitive $Y^{p/s,q/s}$ cells into each other, and so is a \mathbb{Z}_s symmetry.

We now go about finding the rephasing symmetries associated with such diagrams. First, we will use the fact that the superpotential terms allow us to eliminate the internal lines, and so we may discuss only the phases of the sides of the polygon. For brevity, we will always label the fields on the outside with U_i and the subscript will denote the node that the arrow is pointing away from. The internal lines will be labeled as Z_i for terms entering in cubic superpotential terms and Y_i for terms entering in quartic terms, with the i now denoting the node that the arrow is pointing towards. The node labeled 1 will always precede 3 double arrows (which we are guaranteed to have because we assume at least one τ type cell). With this labeling, we may write down a set of rules that determines the anomaly cancelation condition for an arbitrary node in the diagram. The rule is based purely on the previous 2 and following two fields on the edge of the polygon. The previous two fields are either two double lines (D, D), a single line and then a double line (S, D) or a double and then a single line (D, S).

The fields coming after the node fall into the same categories. From this information, the anomaly condition at the m^{th} node

$$\begin{array}{ccccccc}
& U_{m-2} & & U_{m-1} & & U_m & & U_{m+1} \\
\bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \\
m-2 & & m-1 & & m & & m+1 & & m+2 \\
S & \rhd & S & \rhd & S & \rhd & S & \rhd & \\
\text{or} & & \text{or} & & \text{or} & & \text{or} & & \\
D & \rhd\rhd & D & \rhd\rhd & D & \rhd\rhd & D & \rhd\rhd & .
\end{array}$$

is given by

$$\left(\left\{ \begin{array}{l} (D, D) \mapsto \frac{1}{u_{m-2}} \\ (S, D) \mapsto \frac{1}{u_{m-3}u_{m-2}} \\ (D, S) \mapsto 1 \end{array} \right\} \times u_{m-1}u_m \times \left\{ \begin{array}{l} (D, D) \mapsto \frac{1}{u_{m+2}} \\ (S, D) \mapsto 1 \\ (D, S) \mapsto \frac{1}{u_{m+1}u_{m+2}} \end{array} \right\} \right)^N = 1 \quad (2.50)$$

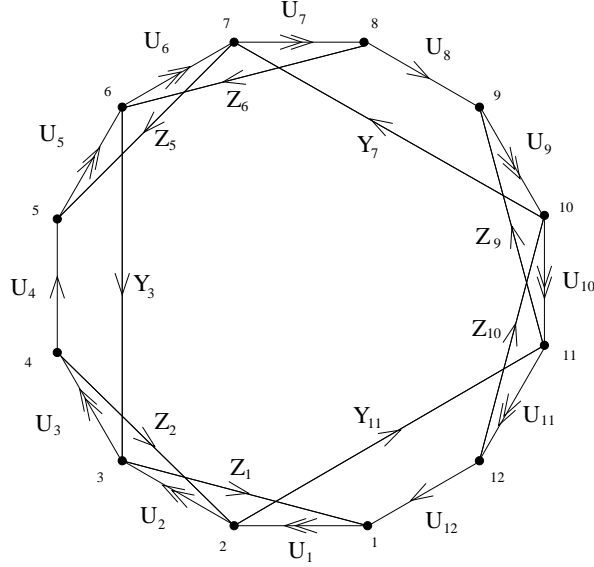
In addition to these, we want that the scalings are naturally associated with a member of the center of the gauge group, and so we require that

$$\prod u_i = 1. \quad (2.51)$$

Now, to fix the B transformation, we set $a_1 = 1, a_2 = \omega, a_3 = \omega^2$. We take the anomaly conditions from node 3 and forward to $2p - 2$, and require that they are trivially satisfied (we mean that the quantity appearing in (2.50) is 1 before raising to the N^{th} power). Taking into account enough equations, a_1, a_2 and a_3 will determine a_4 , and so on up to a_{2p-1} . The quantity a_{2p-1} is then determined by (2.51). The remaining 4 equations are “boundary” terms, involved in consistently “gluing” the ends together. They will give conditions of the form $\omega^{a_i \times N} = 1$ for $i = 1, 2, 2p, 2p - 1$. The a_i should have common divisor of s where again $\text{GCD}(p, q) = s$. This gives us the requirement that $\omega^{sN} = 1$. Rasing this B operation to the s^{th} power automatically is in the center of the gauge group.

To determine the C operation, first note that the A operation naturally associates sets of fields that transform into each other. In particular U_i is associated with $U_{i+2p/a}$, which is associated with $U_{i+4p/a}$ etcetera. From this association, we may associate their scalings. They should satisfy $u_{i+4p/s} = \omega^{k_i} u_{i+2p/s} = \omega^{2k_i} u_i$. We may read off k_1, k_2 and k_3 from the first set of fields. We now repeat the procedure above, only we seed with $a_1 = \omega^{k_1}, a_2 = \omega^{k_2}, a_3 = \omega^{k_3}$. Again, the fact that C is an s^{th} root of a member of the center of the gauge group is true by construction.

To be more concrete, we work an example. Namely, we will work out A , B and C transformations for the quiver of $Y^{6,3}$:



First, the A transformation is easy to pick out as being

$$\begin{aligned}
 A : \quad & (1, 5, 9) \mapsto (9, 1, 5), \\
 & (2, 6, 10) \mapsto (10, 2, 6) \\
 & (3, 7, 11) \mapsto (11, 3, 7) \\
 & (4, 8, 12) \mapsto (12, 4, 8).
 \end{aligned} \tag{2.52}$$

Next, the anomaly cancellation conditions are

$$\begin{aligned}
 \left(\frac{u_{12}u_1}{u_2} \right)^N &= 1, & \left(\frac{u_1u_2}{u_{11}u_{12}u_3} \right)^N &= 1, \\
 \left(\frac{u_2u_3}{u_1u_4u_5} \right)^N &= 1, & \left(\frac{u_3u_4}{u_2} \right)^N &= 1, \\
 \left(\frac{u_4u_5}{u_6} \right)^N &= 1, & \left(\frac{u_5u_6}{u_3u_4u_7} \right)^N &= 1, \\
 \left(\frac{u_6u_7}{u_5u_8u_9} \right)^N &= 1, & \left(\frac{u_7u_8}{u_6} \right)^N &= 1, \\
 \left(\frac{u_8u_9}{u_{10}} \right)^N &= 1, & \left(\frac{u_9u_{10}}{u_7u_8u_{11}} \right)^N &= 1, \\
 \left(\frac{u_{10}u_{11}}{u_9u_{12}u_1} \right)^N &= 1, & \left(\frac{u_{11}u_{12}}{u_{10}} \right)^N &= 1.
 \end{aligned} \tag{2.53}$$

We begin solving the equations as prescribed above, solving the 2nd through 5th line trivially using the prescription defined above. The solution for the B transformation is

$$B : U_i \mapsto u_i U_i, \quad Z_i \mapsto z_i Z_i, \quad Y_i \mapsto y_i Y_i \quad (2.54)$$

with

$$\begin{aligned} u_1 &= 1 & u_5 &= \omega^4 & u_9 &= \omega^8 \\ u_2 &= \omega & u_6 &= \omega^3 & u_{10} &= \omega^5 \\ u_3 &= \omega^2 & u_7 &= \omega^6 & u_{11} &= \omega^{10} \\ u_4 &= \omega^{-1} & u_8 &= \omega^{-3} & u_{12} &= \omega^{-35} \end{aligned} \quad (2.55)$$

and

$$\begin{aligned} z_1 &= \omega^{-1} & z_5 &= \omega^{-7} & z_9 &= \omega^{-13} \\ z_2 &= \omega^{-3} & z_6 &= \omega^{-9} & z_{10} &= \omega^{-15} \\ y_3 &= \omega^{-5} & y_7 &= \omega^{-11} & y_{11} &= \omega^{25} \end{aligned} \quad (2.56)$$

The set of z and y variables are simply read off from the superpotential constraints. We find $k_1 = 4, k_2 = 2, k_3 = 4$. Repeating the above procedure, we find that

$$C : U_i \mapsto u_i U_i, \quad Z_i \mapsto z_i Z_i, \quad Y_i \mapsto y_i Y_i \quad (2.57)$$

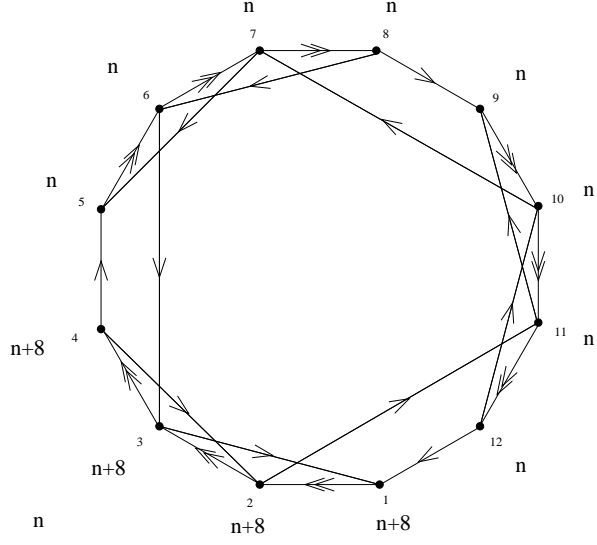
with

$$\begin{aligned} u_1 &= \omega^4 & u_5 &= \omega^4 & u_9 &= \omega^4 \\ u_2 &= \omega^2 & u_6 &= \omega^2 & u_{10} &= \omega^2 \\ u_3 &= \omega^4 & u_7 &= \omega^4 & u_{11} &= \omega^4 \\ u_4 &= \omega^{-2} & u_8 &= \omega^{-2} & u_{12} &= \omega^{-26} \end{aligned} \quad (2.58)$$

and

$$\begin{aligned} z_1 &= \omega^{-6} & z_5 &= \omega^{-6} & z_9 &= \omega^{-6} \\ z_2 &= \omega^{-6} & z_6 &= \omega^{-6} & z_{10} &= \omega^{-6} \\ y_3 &= \omega^{-6} & y_7 &= \omega^{-6} & y_{11} &= \omega^{18} \end{aligned} \quad (2.59)$$

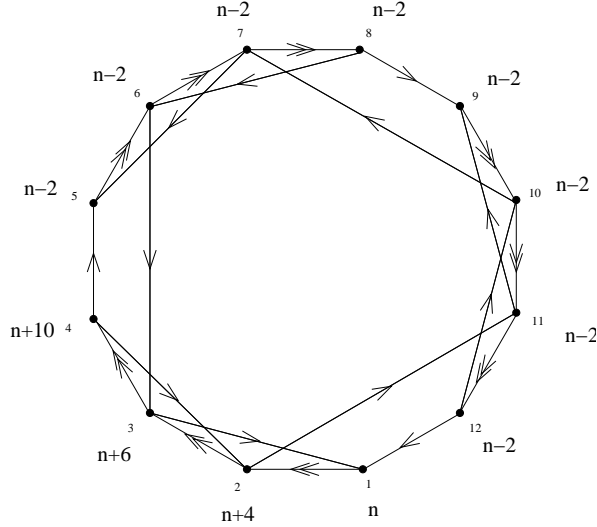
The final requirements from the first and last lines of (2.53) are satisfied for $\omega^{3N} = 1$ (actually, one could have the less restrictive $\omega^{6N} = 1$, however, we wish to find objects such that they are one when raised to the 3rd power, so that the order matches that of A). One may check that A and C commute up to the center of the gauge group generated by



One may also check that

$$AB = BAC \quad (2.60)$$

up to the center of the gauge group associated with



Again, in these diagrams one takes the center of the gauge group and rotates by ω^{3n_i} and then the charges of the fields under the $SU(N)$ determines the rephasing of the fields. We also find that the above operations satisfy $A^3 = B^3 = C^3 = 1$ up to an element in the center of the gauge group.

3 D-brane interpretation

As suggested in [2] the field theory results can be matched with states in the dual string theory. Namely, one can think of the above operators A, B and C as the operators counting the number of fundamental strings and D-strings wrapped around a 1-cycle and C as the number of D3 branes wrapped on that cycle. Alternatively, one can think of A and B as the operators counting the number of NS5 branes and D5 branes wrapping a 3-cycles and C the operator counting the number of D3 branes wrapping a 1-cycle.

Another powerful point of view discussed in [2] and more recently in [14] identifies the A and B symmetries with Wilson and t' Hooft loops. The idea is that a wrapped D3 branes supports a $U(1)$ gauge field in its worldvolume. This gauge field basically carries fundamental and D string numbers in the guise of eigenvalues of Wilson and t' Hooft loops. This point of view is powerful because, at low energies, it reduces the problem to a $U(1)$ gauge field in a nontrivial worldvolume, much about this situation has been discussed in [4]. One expects that topological arguments are then independent of the low energy limit.

Returning to the string theory interpretation. The A symmetry, being perturbative, is naturally identified with perturbative states in the string theory, i. e., the fundamental string. An important piece of evidence in identifying the string states comes from the fact that Heisenberg groups admit an action of $SL(2, \mathbb{Z})$ which we interpret as a symmetry of type IIB string theory. Under an $SL(2, \mathbb{Z})$ transformation given by the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (3.1)$$

the operators transform as

$$A \mapsto A^a B^b, \quad B \mapsto A^c B^d, \quad C \mapsto C. \quad (3.2)$$

One important check in the identification of the string theory states is that the number of states coming from wrapped branes and wrapped fundamental strings is given by the existence of the corresponding cycles in the dual string theory. For example, for orbifolds of \mathbb{C}^3 the relevant string theory is $AdS_5 \times S^5/\mathbb{Z}_q$. The calculation of the homology groups is different from the one presented in the appendix of [2]. The main

difficulty is that the resulting space is not a Lens space due to the fact that the orbifold action is not compatible with the action of the Hopf fibration $S^1 \longrightarrow S^5 \longrightarrow \mathbb{CP}^2$. To see this recall that the orbifold action is inherited from the action on \mathbb{C}^3 given as $(z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, \omega^{-2} z_3)$ where ω is a q -th root of unity. This action was used in section two. On the other hand, the Lens space is defined as the quotient of S^5 by the action of $\mathbb{Z}_q : (z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, \omega z_3)$ which is, of course compatible with the Hopf fibration. Note that only for $q = 3$ the gauge theory orbifold is isomorphic to the orbifold in the definition of Lens space. Nevertheless, using the Leray spectral sequence one finds that, the relevant torsion terms are given by

$$\begin{aligned} H_1(S^5/\mathbb{Z}_q) &= \mathbb{Z}_q, \\ H_3(S^5/\mathbb{Z}_q) &= \mathbb{Z}_q. \end{aligned} \tag{3.3}$$

This is in agreement with our discussion of orbifolds of $\mathcal{N} = 4$ SYM theory in section two.

The calculation of homology groups in the case of the $T^{1,1}$ and, in general for $Y^{p,q}$ spaces is less straightforward. However, here too there are various approaches. In one approach it is convenient to consider the cone over these spaces which are toric varieties for which the relevant properties are known. Alternatively [10], one attacks the calculation by viewing these spaces as $U(1)$ bundles over some Kähler-Einstein base. The relevant results are compiled in [9, 10]. The most relevant statement follows from Lerman [15] and states that the fundamental group of $Y^{p,q}$ manifolds is

$$\pi_1(Y^{p,q}) = \mathbb{Z}_h, \tag{3.4}$$

where h is the highest common factor of (p, q) . Further we use, that the fundamental group is isomorphic to the first homology group which coincides with the third homology group. Note that this is precisely the fact that was exploited in constructing the A transformation in section two.

A correction for $SL(2, \mathbb{Z})$ in the presence of flux?

We see that the $SL(2, \mathbb{Z})$ action, as written above, is broken in the general case. The reason is its incompatibility with the cyclic property of the group elements. To understand the problem note that if we perform an $SL(2, \mathbb{Z})$ transformation given by

the matrix (3.1) then the cyclic property of, say, A is affected:

$$\begin{aligned} A^q &\mapsto \underbrace{A^a B^b \dots A^a B^b}_{q\text{-times}} \\ &= A^{qa} B^{qb} C^{-\frac{1}{2}q(q+1)ab}. \end{aligned} \quad (3.5)$$

Given that $A^q = B^q = C^q = 1$ this is not a problem if q is odd. Namely, for $q = 2r + 1$ odd, we have that after the transformation $A^q \mapsto C^{q(-(r+1)ab)}$ which is the unity transformation and therefore cyclicity is preserved. However for $q = 2r$ even we see that $A^q \mapsto C^{q(-\frac{1}{2}(2r+1)ab)}$ which is not necessarily the identity transformation. The failure, however, is restricted to a square root of minus one.

One can attempt to remedy the situation by suggesting that the $SL(2, \mathbb{Z})$ action is

$$M : \quad A \mapsto A^a B^b C^{ab/2}, \quad B \mapsto A^c B^d C^{cd/2}, \quad C \mapsto C. \quad (3.6)$$

Note, since C commutes with A and B this is like simply multiplying by some c -number.

4 Conclusions

In this note we have explicitly demonstrated the existence of finite Heisenberg groups realized as discrete transformation in a large class of quiver gauge theories. We believe our work has interest implications for our understanding of the D-branes.

Given the current state of string theory technology, understanding branes in backgrounds with Ramond-Ramond fluxes is beyond our means. In this sense the AdS/CFT provides a unique opportunity to study the nature of D-branes. In this case, we established indirectly that in backgrounds with nontrivial RR flux, if the space has torsion classes in the homology groups, the operators whose eigenvalues determine the number of branes wrapping the corresponding cycles satisfy a finite Heisenberg group.

There are various issues that remain to be explored and we hope to return to some of them in the near future. First there is the question of whether most quiver gauge theories admit such structure, in particular if all toric quivers admit such construction. As we suspect a necessary condition is given by the existence of torsion in the first and third homology classes of the horizon geometry. Another interesting question, which we plan to explore is the relationship between the finite Heisenberg groups and Seiberg duality in the quiver. It would also be interesting to consider nonconformal situations. Naively, given the conditions we impose in the construction of the operators A, B and

C , one should not expect such symmetry to survive the nonconformal limit. However, if this is a general property of D-branes there should be an action of a finite Heisenberg group. More generally one wonders if noncommutativity is a property of D-branes in backgrounds with RR flux how could one approach a similar calculation for IIA string theory.

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